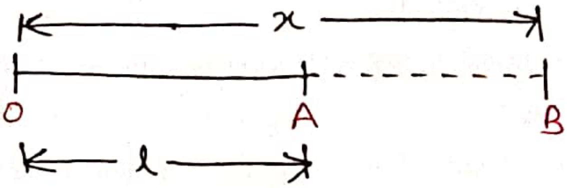
Hooke's law

— The tension of an elastic string is proportional to the extension of the string beyond its natural Length.



Let the natural length of the string OA be 'l' and let $OB = x$ be its stretched length.
If the tension in the string be T then by Hooke's law

$$T \propto \frac{\text{Extension in length}}{\text{Natural length}}$$

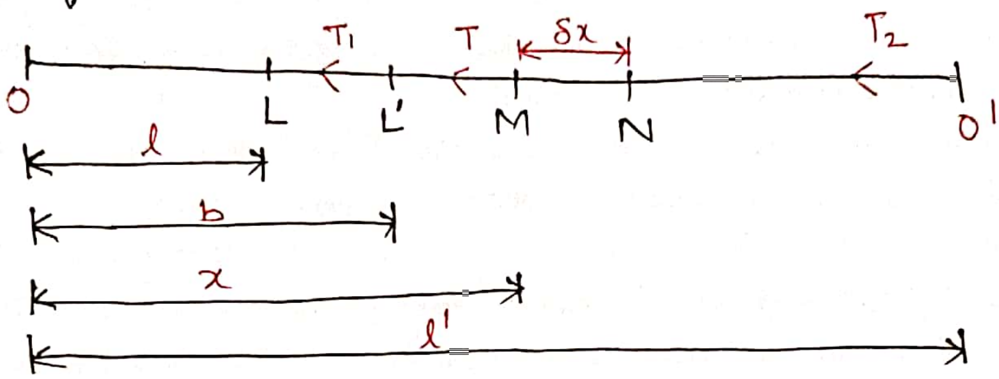
$$\boxed{T = \frac{\lambda (x - l)}{l}}$$

The constant λ is called modulus of elasticity of the string.

Theorem:

Prove that the work done against the tension in stretching a light elastic string is equal to the product of its extension and the mean of the initial and final tensions.

Proof:



Let OL be a uniform light elastic string of natural length 'l' and let the end O be fixed.

Let the string be stretched to L'

where, $OL' = b$.

The string OL' is again stretched from L' to O' such that $OO' = l'$

We have to find the work done against the tension in stretching the string from L' to O'.

Let λ be the modulus of elasticity of the string and let T, T_1, T_2 be the tensions at M, L' and O' resp.

Also, let $OM = x$

$ON = x + \delta x$

where, δx is very small in comparision with x, so that the tension of the string in between M and N is approx. the same.

$$MN = ON - OM$$
$$= x + \delta x - x$$
$$= \delta x$$

If the string be stretched to the length x then by Hooke's law

Tension at M,

$$T = \frac{\lambda(x - l)}{l} \quad \text{———} \quad (1)$$

Also, the tension at L', i.e. the initial Tension will be

$$T_1 = \frac{\lambda(b - l)}{l} \quad \text{———} \quad (2)$$

and Tension at O', i.e. final Tension will be

$$T_2 = \frac{\lambda(l' - l)}{l} \quad \text{———} \quad (3).$$

Extension produced $= L'O'$

$$= OO' - OL'$$
$$= l' - b$$

Therefore, work done against the tension in stretching the string from M to N

$$= \frac{\lambda (x-l)}{l} \cdot \delta x$$

Hence, the total work done in stretching the light elastic string from an initial position L' to another position O'

$$= \int_b^{l'} \frac{\lambda (x-l)}{l} \, dx$$

$$= \frac{\lambda}{l} \int_b^{l'} (x-l) \, dx$$

$$= \frac{\lambda}{l} \left[\frac{(x-l)^2}{2} \right]_b^{l'}$$

$$= \frac{\lambda}{l} \left[\frac{(l'-l)^2}{2} - \frac{(b-l)^2}{2} \right]$$

$$= \frac{\lambda}{2l} \left[(l'-l)^2 - (b-l)^2 \right]$$

$$= \frac{\lambda}{2l} \left[(l'-l+b-l)(l'-l-b+l) \right]$$

$$= \frac{\lambda}{2l} \left[(l'+b)(l'-b) \right].$$

$$= \frac{(l'-b)\lambda}{2} \left[\frac{l'+b}{l} \right]$$

$$= \frac{(l'-b)\lambda}{2} \left[\frac{l'-l}{l} + \frac{b-l}{l} \right].$$

$$= \frac{(l'-b)}{2} \left[\frac{\lambda (l'-l)}{l} + \frac{\lambda (b-l)}{l} \right].$$

$$= \frac{(l'-b)}{2} \left[T_2 + T_1 \right]$$

$$= (l' - b)\frac{(T_1 + T_2)}{2}$$

= Extension produced × Mean of Initial and final Tension.